**Title:** Alienation and the Stability Problem

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**Citation style:** Sikorska Justyna. (2020). Alienation and the Stability Problem. “Results in Mathematics” (Vol. 75 (2020), Art. No. 98), doi 10.1007/s00025-020-01216-1

Results Math (2020) 75:98
*Online First*

https://doi.org/10.1007/s00025-020-01216-1

**Results in Mathematics**

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# Alienation and the Stability Problem

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*Dedicated to the memory of Professor János Aczél*

**Abstract.** Starting from the inequality

$$|f(x+y)-f(x)-f(y)+g(x+y)-g(x)g(y)| \leqslant \varepsilon, \quad x,y \in S,$$

where $f$ is a complex valued function defined on a monoid $S$, we deal with two problems: the stability problem and the problem of alienation of the approximate additivity condition from the condition of approximate exponentiality.

**Mathematics Subject Classification.** 39B82, 39B62, 39B52.

**Keywords.** Additivity, exponentiality, quadratic equation, ring homomorphism, normed algebra, stability, alienation.

## 1. Introduction

Since 1988, when J. Dhombres published his paper [5], many results concerning alienation of functional equations have appeared. Shortly speaking, given two functional equations $E_1(f)=0$ and $E_2(g)=0$ for two (possibly the same) functions $f$ and $g$, we add the equations side by side obtaining

$$E_1(f)+E_2(g)=0, \tag{1}$$

and we ask, one can think—hopelessly, whether from the above equation the starting conditions follow, that is,

$$\begin{cases} E_1(f)=0 \\ E_2(g)=0. \end{cases} \tag{2}$$

Published online: 08 June 2020

Surprisingly, in many situations we observe such effect. We say then that equations in (2) are *alien* to each other. For a number of references on the subject, see a survey by R. Ger and M. Sablik [9].

Similarly, starting from two inequalities (for functions with values in normed spaces)

$$\|E_1(f)\| \leqslant \varepsilon_1 \qquad \text{and} \qquad \|E_2(g)\| \leqslant \varepsilon_2, \tag{3}$$

it follows that

$$\|E_1(f) + E_2(g)\| \leqslant \varepsilon \tag{4}$$

with $\varepsilon = \varepsilon_1 + \varepsilon_2$. Now, conversely, given $E_1$, $E_2$, assume that functions $f$, $g$ satisfy (4). Having in mind typical stability results (see, e.g., [11]) and the alienation phenomenon mentioned above, our considerations are going in two directions. First we ask whether there exist functions $\widetilde{f}$ and $\widetilde{g}$ satisfying (1), i.e., $E_1(\widetilde{f}) + E_2(\widetilde{g}) = 0$, and such that functions $f$, $\widetilde{f}$ and $g$, $\widetilde{g}$ are in a sense close. The latter direction is determined by the alienation considerations: we ask whether there exist nonnegative $\varepsilon_1$, $\varepsilon_2$ such that (3) holds. In the first case we say that equation (1) is *stable* and in the second—we will say that conditions in (3) are *alien* to each other. Even though in the literature one can find results treating inequalities of the form (4), the research concerns stability. The alienation point of view seems to be new in this situation.

In 1949, D.G. Bourgin [4] was studying the following system of inequalities

$$\begin{cases} \|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon_1 \\ \|f(xy) - f(x)f(y)\| \leqslant \varepsilon_2 \end{cases} \tag{5}$$

for $f$ acting between two Banach algebras with units. Assuming the surjectivity of $f$, he obtained that $f$ had to be a ring homomorphism, that is, $f$ satisfies both $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$. R. Badora ([3]) generalized this result getting rid of the surjectivity assumption.

From (5) it follows that

$$\|f(x+y) + f(xy) - f(x) - f(y) - f(x)f(y)\| \leqslant \varepsilon. \tag{6}$$

In [6], R. Ger studied inequality (6) assuming that

> $f$ acts from a ring $\mathcal{R}$ with a unit 1 into a commutative Banach algebra $\mathcal{A}$ with a unit $e$, and $f(0) = 0$, $f(1) = e$, $f(2) = 2e$.

He derived that either there exist nonzero elements $a \in \mathcal{A}$, $r \in \mathcal{R}$ such that the map $\mathcal{R} \ni x \mapsto af(rx) \in \mathcal{A}$ is bounded, or $f$ is a ring homomorphism. In the case $\mathcal{R}$ is a field and $\mathcal{A} = \mathbb{C}$, it means that either $f$ is bounded or it satisfies the equation

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y), \tag{7}$$

which sometimes is called as a *superstability* phenomenon. At the same time (5) holds, which means that inequalities in (5) are alien from each other. In fact, as it was shown, if $f$ is unbounded, then $\varepsilon_1 = \varepsilon_2 = 0$.

In [2], M. Adam was studying the stability of the equation

$$f(x+y)+f(x-y)+g(x+y)=2f(x)+2f(y)+g(x)+g(y)$$

for functions acting on a 2-divisible abelian group $G$ and with values in a Banach space. He proved that there exist a unique additive function $a$ and a unique quadratic function $q$ such that $\|f(x)-f(0)-a(x)\|\leqslant 24\varepsilon$ and $\|g(x)-g(0)-q(x)\|\leqslant 13\varepsilon$ for all $x\in G$. Therefore, with $\widetilde{f}=a$ and $\widetilde{g}=q$ we have $\|f(x)-\widetilde{f}(x)\|\leqslant\|f(0)\|+24\varepsilon$ and $\|g(x)-\widetilde{g}(x)\|\leqslant\|g(0)\|+13\varepsilon$ for all $x\in G$. This means that if $E_1$ denotes the Cauchy difference and $E_2$ denotes the quadratic difference, then $E_1(f)$ and $E_2(g)$ are bounded, i.e., inequalities in (3) are alien from each other. At the same time, from [2] and [8] we know that the quadratic equation and the additive Cauchy equation are alien to each other (only) up to a constant.

In what follows we are interested in two Cauchy equations: the additive and the exponential ones on some monoid $S$

$$\begin{cases} f(x+y)=f(x)+f(y), & x,y\in S,\\ g(x+y)=g(x)g(y), & x,y\in S,\end{cases} \tag{8}$$

and their sum

$$f(x+y)+g(x+y)=f(x)+f(y)+g(x)g(y),\quad x,y\in S. \tag{9}$$

The solutions of (9) were obtained by R. Ger in [7]. It turns out that without any additional assumptions, the equations in (8) are not alien to each other (cf., [7, Section 4]).

Given $\varepsilon\geqslant 0$ and $f,g\colon S\to\mathbb{C}$, we consider now the following inequality

$$|f(x+y)-f(x)-f(y)+g(x+y)-g(x)g(y)|\leqslant\varepsilon,\quad x,y\in S,$$

and we ask whether (9) is stable. At the same time, we are interested whether there exist $\varepsilon_1,\varepsilon_2\geqslant 0$ such that

$$|f(x+y)-f(x)-f(y)|\leqslant\varepsilon_1,\quad x,y\in S, \tag{10}$$

and

$$|g(x+y)-g(x)g(y)|\leqslant\varepsilon_2,\quad x,y\in S, \tag{11}$$

what would mean that inequalities (10) and (11) are alien to each other.

## 2. Main Results

In what follows we present the main result of the paper for functions with complex values.

**Theorem 1.** *Let $(S,+)$ be an abelian monoid. Given an $\varepsilon\geqslant 0$, let $f,g\colon S\to\mathbb{C}$ satisfy*

$$|f(x+y)+g(x+y)-f(x)-f(y)-g(x)g(y)|\leqslant\varepsilon,\quad x,y\in S. \tag{12}$$

*Then either*

*(i) $g$ is bounded and there exists an additive function $a\colon S \to \mathbb{C}$ such that $f - a$ is bounded,*

*or*

*(ii) $g$ is unbounded and there exists a function $\widetilde{f}\colon S \to \mathbb{C}$ such that*

$$\widetilde{f}(x+y) + g(x+y) = \widetilde{f}(x) + \widetilde{f}(y) + g(x)g(y), \quad x, y \in S,$$

*and*

$$|f(x) - \widetilde{f}(x)| \leqslant \varepsilon, \quad x \in S.$$

*Proof.* Observe first that if $g$ is bounded, say $|g(x)| \leqslant M$ for all $x \in S$ and some $M > 0$, then

$$|f(x+y) - f(x) - f(y)| \leqslant \varepsilon + M + M^2, \quad x, y \in S,$$

and by [10], there exists an additive function $a\colon S \to \mathbb{C}$ such that

$$|f(x) - a(x)| \leqslant \varepsilon + M + M^2, \quad x \in S.$$

Assume now that $g$ is unbounded. On account of (12), for all $x, y, z \in S$ we have

$$\begin{gathered}
|f(x+y+z) + g(x+y+z) - f(x) - f(y+z) - g(x)g(y+z)| \leqslant \varepsilon,\\
|f(y+z) + g(y+z) - f(y) - f(z) - g(y)g(z)| \leqslant \varepsilon,\\
|-f(x+y+z) - g(x+y+z) + f(x+y) + f(z) + g(x+y)g(z)| \leqslant \varepsilon,\\
|-f(x+y) - g(x+y) + f(x) + f(y) + g(x)g(y)| \leqslant \varepsilon.
\end{gathered}$$

From the above inequalities we obtain

$$\big|g(x{+}y)g(z) - g(x{+}y) + g(x)g(y) - g(x)g(y{+}z) + g(y{+}z) - g(y)g(z)\big| \leqslant 4\varepsilon,$$

for all $x, y, z \in S$.

Consider now function $h\colon S \to \mathbb{C}$ defined by $h(x) := g(x) - 1$ for all $x \in S$. Then

$$\big|h(x+y)h(z) + h(x)h(y) - h(x)h(y+z) - h(y)h(z)\big| \leqslant 4\varepsilon, \quad x, y, z \in S,$$

that is,

$$\big|\big(h(x+y) - h(y)\big)h(z) - h(x)\big(h(y+z) - h(y)\big)\big| \leqslant 4\varepsilon, \quad x, y, z \in S,$$

and

$$\left|h(x+y) - h(y) - h(x)\frac{h(y+z) - h(y)}{h(z)}\right| \leqslant \frac{4\varepsilon}{|h(z)|}, \tag{13}$$

for all $x, y, z \in S$ such that $h(z) \neq 0$.

Since $g$ is unbounded, so is $h$. Take a sequence $(z_n)_{n\in\mathbb{N}}$ of elements of $S$ such that $h(z_n) \neq 0$ and $\lim_{n\to\infty} |h(z_n)| = \infty$. Then

$$\lim_{n\to\infty} \frac{h(y+z_n)-h(y)}{h(z_n)} = \frac{h(x+y)-h(y)}{h(x)}, \tag{14}$$

for all $x, y \in S$ such that $h(x) \neq 0$. The above limit exists and does not depend on the choice of the sequence $(z_n)_{n\in\mathbb{N}}$. Hence, there exists a function $\varphi\colon S \to \mathbb{C}$ such that

$$h(x+y) = h(x)\varphi(y) + h(y), \quad x, y \in S. \tag{15}$$

Indeed, if $h(x) \neq 0$ then the above equality we get from (14). Otherwise, it follows immediately from (13) (and the unboundedness of $h$).

On account of [1, Theorem 1, p. 242], we determine the general solution of (15). Namely, (i) $h$ is additive and $\varphi(x) \equiv 1$, or (ii) $h(x) = c[1-e(x)]$ and $\varphi(x) = e(x)$ for all $x \in S$, where $e$ is exponential and $c$ is a constant, or (iii) $h$ is constant which contradicts to our assumption that $g$ and therefore $h$, are unbounded. We consider now each of the first two cases. In fact, we are not interested in the solution $\varphi$.

(i) Assume $h$ is additive (let us call it $a$). Substituting $g(x) = a(x)+1$ to (12) we obtain

$$|f(x+y) - f(x) - f(y) - a(x)a(y)| \leqslant \varepsilon, \quad x, y \in S.$$

Consider now $\psi := f - \frac{1}{2}a^2$. Then

$$|\psi(x+y) - \psi(x) - \psi(y)| \leqslant \varepsilon, \quad x, y \in S.$$

There exists (cf. [10]) an additive function $b\colon S \to \mathbb{C}$ such that

$$|\psi(x) - b(x)| \leqslant \varepsilon, \quad x \in S,$$

that is,

$$\left|f(x) - \tfrac{1}{2}a(x)^2 - b(x)\right| \leqslant \varepsilon, \quad x \in S.$$

Define $\widetilde{f}(x) := \frac{1}{2}a(x)^2 + b(x)$ for all $x \in S$. Then

$$\widetilde{f}(x+y) - \widetilde{f}(x) - \widetilde{f}(y) + g(x+y) - g(x)g(y) = 0, \quad x, y \in S.$$

Moreover,

$$|f(x) - \widetilde{f}(x)| \leqslant \varepsilon, \quad x \in S.$$

(ii) Assume $h(x) = c[1-e(x)]$, $x \in S$, where $e$ is a nonzero exponential function (case $e(x) \equiv 0$ is excluded by the unboundedness assumption) and $c$ is a constant. Then $g(x) = 1 + c[1-e(x)]$ for all $x \in S$. We put this form to (12):

$$\begin{aligned}\big|f(x+y) - f(x) - f(y) + 1 + c[1-e(x+y)]\\ -\big(1+c[1-e(x)]\big)\big(1+c[1-e(y)]\big)\big| \leqslant \varepsilon,\end{aligned}$$

that is, for all $x, y \in S$,

$$\begin{aligned}\big|f(x+y) - f(x) - f(y) + 1 + c - ce(x+y) - 1 - c + ce(y)\\ - c - c^2 + c^2e(y) + ce(x) + c^2e(x) - c^2e(x)e(y)\big| \leqslant \varepsilon,\end{aligned}$$

and

$$\begin{aligned}\big|\big(f(x+y) - c(c+1)e(x+y)\big) - \big(f(x) - c(c+1)e(x)\big)\\ - \big(f(y) - c(c+1)e(y)\big) - c(c+1)\big| \leqslant \varepsilon,\end{aligned}$$

for all $x, y \in S$.

Define $\psi(x) := f(x) - c(c+1)\big(e(x) - 1\big)$ for all $x \in S$. Then the above inequality takes the form

$$|\psi(x+y) - \psi(x) - \psi(y)| \leqslant \varepsilon, \quad x, y \in S.$$

There exists an additive function $a\colon S \to \mathbb{C}$ such that

$$|\psi(x) - a(x)| \leqslant \varepsilon, \quad x \in S,$$

which means that

$$\big|f(x) - c(c+1)\big(e(x) - 1\big) - a(x)\big| \leqslant \varepsilon, \quad x \in S.$$

Define $\widetilde{f}(x) := a(x) + c(c+1)\big(e(x) - 1\big)$ for all $x \in S$. Then

$$\widetilde{f}(x+y) - \widetilde{f}(x) - \widetilde{f}(y) + g(x+y) - g(x)g(y) = 0, \quad x, y \in S.$$

Moreover,

$$|f(x) - \widetilde{f}(x)| \leqslant \varepsilon, \quad x \in S$$

□

*Remark 1.* It follows from Theorem 1 (and its proof) that in the case $g$ is bounded, inequalities (10) and (11) are alien to each other. Indeed, from (12) we obtain

$$\begin{cases} |f(x+y) - f(x) - f(y)| \leqslant \varepsilon_1, & x, y \in S, \\ |g(x+y) - g(x)g(y)| \leqslant \varepsilon_2, & x, y \in S, \end{cases}$$

with $\varepsilon_1 = \varepsilon + M + M^2$ and $\varepsilon_2 = M + M^2$.

If $g$ is unbounded, in general, the mentioned inequalities are not alien to each other. It would happen only (see the proof of Theorem 1) in the situations

(i) $a(x) \equiv 0$ (in fact, whenever $a$ is bounded), but then $g(x) \equiv 1$ which contradicts to the assumption, or

(ii) $c(c+1)\big(e(x) - 1\big) \equiv 0$, that is, whenever $c = 0$ or $c = -1$ or $e(x) \equiv 1$. Only the case $c = -1$ fulfils our requirements; then $\widetilde{f} = a$ and $g = e$.

The computations provided in the proof of Theorem 1 cannot be repeated for functions with values in normed algebras. In what follows we present only a partial result in this direction.

**Theorem 2.** *Let $(S,+)$ be a monoid and $(\mathcal{A},\|\cdot\|)$ be a unital normed algebra. Given an $\varepsilon \geqslant 0$, let $f,g\colon S \to \mathcal{A}$ satisfy*

$$\|f(x+y)-f(x)-f(y)+g(x+y)-g(x)g(y)\| \leqslant \varepsilon, \quad x,y \in S. \tag{16}$$

*Then there exist constants $c,d \in \mathcal{A}$ such that functions $S \ni x \mapsto cg(x)+d \in \mathcal{A}$ and $S \ni x \mapsto g(x)c+d \in \mathcal{A}$ are bounded. More precisely,*

$$\|cg(x)+d\| \leqslant 2\varepsilon \quad \text{and} \quad \|g(x)c+d\| \leqslant 2\varepsilon, \quad x \in S. \tag{17}$$

*Proof.* With $x=y=0$ in (16) we obtain

$$\|-f(0)+g(0)-g(0)^2\| \leqslant \varepsilon,$$

and with $y=0$ we have

$$\|-f(0)+g(x)-g(x)g(0)\| \leqslant \varepsilon, \quad x \in S,$$

which gives

$$\|g(x)-g(0)+g(0)^2-g(x)g(0)\| \leqslant 2\varepsilon, \quad x \in S,$$

and, since $1 \in \mathcal{A}$,

$$\big\|g(x)\big(1-g(0)\big)+g(0)^2-g(0)\big\| \leqslant 2\varepsilon, \quad x \in S.$$

Analogously (setting $x:=0$), we get

$$\big\|\big(1-g(0)\big)g(y)+g(0)^2-g(0)\big\| \leqslant 2\varepsilon, \quad y \in S.$$

This means that we have our assertion with $c:=1-g(0)$ and $d:=g(0)^2-g(0)$.

Surely, the above result gives no information about $g$ in the case $c=0$. Moreover, if $\mathcal{A}$ is a field and $c \neq 0$, it tells nothing else but $g$ is bounded.



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Received: February 12, 2020.
Accepted: May 4, 2020.